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Universality of the Local Eigenvalue Statistics for a Class of Unitary Invariant Random Matrix Ensembles

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This paper is devoted to the rigorous proof of the universality conjecture of random matrix theory, according to which the limiting eigenvalue statistics of $n \times n$ random matrices within spectral intervals of $O(n^{-1})$ is determined by the type of matrix (real symmetric, Hermitian, or quaternion real) and by the density of states. We prove this conjecture for a certain class of the Hermitian matrix ensembles that arise in the quantum field theory and have the unitary invariant distribution defined by a certain function (the potential in the quantum field theory) satisfying some regularity conditions.

KEY WORDS: Random matrices; local asymptotic regime; universality conjecture; orthogonal polynomial technique.

1. INTRODUCTION. PROBLEM AND RESULTS

The random matrix theory (RMT) has been extensively developed and used in a number of areas of theoretical and mathematical physics. In particular the theory provides a quite satisfactory description of fluctuations in the spectra of complex quantum systems such as heavy nuclei, small metallic particles, and classically chaotic quantum models. One of the important ingredients of this description is the universality conjecture of the RMT, according to which the local eigenvalue statistics on $n \times n$ random matrices (probabilistic properties of their spectra within intervals of the order of $1/n$) does not depend on a particular ensemble in the limit $n = \infty$ and is completely determined by the invariance group of the ensemble probability distribution. There are three invariance groups

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(orthogonal, unitary, and simplectic) and three respective classes of the random matrix ensembles that model quantum systems possessing, respectively, invariance under time reflection and space rotations. The explicit form of the local eigenvalue statistics in the limit $n=\infty$ for each of these classes was found the 1960s by Wigner, Mehta, Dyson, and others, who introduced and studied the explicitly solvable Gaussian and circular ensembles (see ref. 1 and references therein).

In this paper we consider the technically simplest case of the unitary invariant ensembles. Moreover, we will study the class defined by the density

$$p_n(M)\,dM = Z_n^{-1} \exp\{-n \operatorname{Tr} V(M)\}\,dM \tag{1.1}$$

where M is an $n\times n$ Hermitian matrix,

$$dM = \prod_{j=1}^{n} dM_{jj} \prod_{j<k} d\Im M_{jk}\, d\Re M_{jk}$$

is the "Lebesgue" measure for Hermitian matrices, Z_n is the normalization factor, and $V(\lambda)$ is a real-valued function (see the Theorem below for explicit conditions).

The case $V(\lambda)=\lambda^2/2$ corresponds to the Gaussian unitary ensemble (GUE), which was introduced by Wigner in the 1950s. Ensembles with an arbitrary $V(\lambda)$ were introduced in the 1960s,[2-4] and some particular cases were studied. The new wave of interest in this class of unitary invariant ensembles was caused by quantum field theory, where they arise in the large-n limit of quantum chromodynamics, two-dimensional quantum gravity, and bosonic string theory (see the reviews in refs. 5 and 6). Analogous ensembles are used in condensed matter theory and statistical mechanics of random surfaces.[7,8]

Denote by $p_n(\lambda_1,..., \lambda_n)$ the joint probability density of all eigenvalues, which we assume to be symmetric without loss of generality. Let

$$p_l^{(n)}(\lambda_1,..., \lambda_l) = \int p_n(\lambda_1,..., \lambda_l, \lambda_{l+1},..., \lambda_n)\, d\lambda_{l+1} \cdots d\lambda_n \tag{1.2}$$

be its lth marginal distribution density. The simplest case of $p_1^{(n)}(\lambda_1)$ is of particular interest. Indeed, denote by $\lambda_1^{(n)},..., \lambda_n^{(n)}$ the eigenvalues of a random Hermitian matrix M and set

$$N_n(\Delta) = \frac{1}{n} \sum_{\lambda_l^{(n)}\in\Delta} 1, \qquad \Delta=(a, b) \tag{1.3}$$

This is the normalized counting function (empirical eigenvalue distribution) of the matrix. Then

$$E\{N_n(\Delta)\} = \int_\Delta p_1^{(n)}(\lambda)\,d\lambda \equiv \int_\Delta \rho_n(\lambda)\,d\lambda \tag{1.4}$$

where $E\{\cdots\}$ denotes the expectation with respect to density (1.1).

In a recent paper[9] it was proved that if $V(\lambda)$ is bounded below for all $\lambda \in \mathbf{R}$ and satisfies the conditions

$$V(\lambda) \geqslant (2+\varepsilon)\log|\lambda|, \qquad |\lambda| \geqslant L_0 \tag{1.5a}$$

for some L_0 and

$$|V(\lambda_1) - V(\lambda_2)| \leqslant C(A)\,|\lambda_1 - \lambda_2|^\gamma, \qquad |\lambda_{1,2}| \leqslant A \tag{1.5b}$$

for any $0 < A < \infty$ and some $\gamma > 0$, then $\rho_n(\lambda)$ converges to the limiting density $\rho(\lambda)$ (density of states) in the Hilbert space defined by the norm

$$\left(-\int \log|\lambda-\mu|\,\rho(\lambda)\,\rho(\mu)\,d\lambda\,d\mu\right)^{1/2} \tag{1.6}$$

and $\rho(\lambda)$ can be found from a variational procedure analogous to that known in the mean-field theory of statistical mechanics. Moreover, there exist positive numbers L, L_1 $(L > L_1)$, d and a such that (for $|\lambda| > L$)

$$\begin{aligned} &V(\lambda) - \max_{|\mu| \leqslant L_1} V(\mu) > d \\ &\rho_n(\lambda) \leqslant \exp\{-na[V(\lambda) - \max_{|\mu| \leqslant L_1} V(\mu)]\}, \end{aligned} \tag{1.7}$$

and for any function $\phi(\mu)$ differentiable on $(-L, L)$ which grows not faster than $e^{bV(\mu)}$, $b > 0$, as $|\mu| \to \infty$

$$\left|\int \phi(\mu)\,\rho_n(\mu)\,d\mu - \int \phi(\mu)\,\rho(\mu)\,d\mu\right| \leqslant C\,\|\phi'\|_2^{1/2}\,\|\phi\|_2^{1/2}\,n^{-1/2}\log^{1/2} n \tag{1.8}$$

where symbol $\|\cdots\|_2$ denotes the L_2-norm on $(-L, L)$.

Here and below the symbols C and C_i denote n-independent positive constants that may be different in different formulas.

Now we formulate the universality conjecture following Dyson.[4]

Universality Conjecture. For any n-independent integer l, λ_0 such that $\rho(\lambda_0) \neq 0$, and arbitrary fixed $(x_1,..., x_l) \in \mathbf{R}^l$

$$\begin{aligned} &\lim_{n \to \infty} [\rho_n(\lambda_0)]^{-l} p_l^{(n)} \left(\lambda_0 + \frac{x_1}{n\rho_n(\lambda_0)},..., \lambda_0 + \frac{x_l}{n\rho_n(\lambda_0)} \right) \\ &\qquad = \det \|S(x_1 - x_k)\|_{j,k=1}^{l} \end{aligned} \tag{1.9}$$

where

$$S(x) = \frac{\sin \pi x}{\pi x} \tag{1.10}$$

In other words, the limit in the r.h.s. of (1.9) is the same for all $V(\lambda)$'s in (1.1) (modulo some weak conditions) and all λ_0 that belong to the "bulk" of the spectrum where $\rho(\lambda_0) \neq 0$. Thus the limit (1.9) for arbitrary V has to coincide with the same limit for the archetypal Gaussian case $V(\lambda) = \lambda^2/2$, whose form is given by the r.h.s. of (1.9) and has been known since the early 1960s (see ref. 1 for results and discussions).

In this paper we prove the following result.

Theorem. Assume that the function $V(\lambda)$ satisfies the following conditions:

(i) The condition

$$V(\lambda) \geqslant (2+\varepsilon) \log |\lambda|, \qquad |\lambda| \geqslant L_0 \tag{1.11a}$$

holds for some $L_0 < \infty$ [cf. (1.5)].

(ii) The condition

$$\sup |V'''(\lambda)| \leqslant C(L) < \infty, \qquad |\lambda| \leqslant L \tag{1.11b}$$

holds, where L is defined in (1.7).

(iii) $V'(\lambda)$ exists for all λ and

$$\int (V')^2 (\lambda) \, e^{-kV(\lambda)} \, d\lambda \tag{1.11c}$$

for some $k > 0$.

Then the universality conjecture (1.9) is true uniformly in $(x_1,..., x_l)$ varying on compact sets of $\mathbf{R}^l$.

Remarks. 1. In fact the Theorem is valid without any assumptions on the growth of $V(\lambda)$ provided that conditions (i) and (ii) are valid.

However, our proofs are simpler if $V(\lambda)$ satisfies (iii). To prove the theorem without this condition, we have to restrict from the very beginning all integrals to the finite interval $(-L, L)$ outside of which the estimate (1.7) is valid. The latter allows us to control the remainders and to prove that they vanish exponentially as $n \to \infty$. This approach was used in ref. 9, where a kind of variational argument was applied to study the density of states $\rho(\lambda)$. In our case we can make the same restriction, but since, unlike ref. 9, here we use extensively the orthogonal polynomial technique, which relies strongly on integral relations, the respective estimates are more tedious and require more space. That is why we impose the technical conditions (iii).

2. Denote by $P_l^{(n)}(\lambda)$, $l=0, 1,...,$ orthogonal polynomials on $\mathbf{R}$ associated with the weight

$$w_n(\lambda) = e^{-nV(\lambda)} \tag{1.12}$$

$$\int P_l^{(n)}(\lambda)\, P_m^{(n)}(\lambda)\, e^{-nV(\lambda)}\, d\lambda = \delta_{l,m} \tag{1.13}$$

and by

$$\psi_l^{(n)}(\lambda) = \exp\{-nV(\lambda)/2\}\, P_l^{(n)}(\lambda), \qquad l=0, 1,... \tag{1.14}$$

the respective orthonormal system

$$\int \psi_l^{(n)}(\lambda)\, \psi_m^{(n)}(\lambda)\, d\lambda = \delta_{l,m} \tag{1.15}$$

Then the joint probability density of all the eigenvalues of ensemble (1.1) is[1]

$$p_n(\lambda_1,..., \lambda_n) = \hat{Z}_n^{-1} \prod_{1 \leqslant j<k \leqslant n} (\lambda_i - \lambda_j)^2 \exp\left\{-n \sum_{j=1}^{n} V(\lambda_j)\right\} \tag{1.16}$$

$$= (n!)^{-1} (\det \|\psi_{j-1}(\lambda_k)\|_{j,k=1}^{n})^2 \tag{1.17}$$

and the marginal densities (1.4) are

$$p_l^{(n)}(\lambda_1,..., \lambda_l) = \frac{(n-l)!}{n!} \det \|k_n(\lambda_j, \lambda_k)\|_{j,k=1}^{l} \tag{1.18}$$

where

$$k_n(\lambda, \mu) = \sum_{l=0}^{n-1} \psi_l^{(n)}(\lambda)\, \psi_l^{(n)}(\mu) \tag{1.19}$$

is known as the reproducing kernel of the system (1.14). In particular,

$$\rho_n(\lambda) \equiv p_1^{(n)}(\lambda) = K_n(\lambda, \lambda) \tag{1.20}$$

where

$$K_n(\lambda, \mu) = n^{-1} k_n(\lambda, \mu) \tag{1.21}$$

In view of (1.18) the proof of the universality conjecture (1.9) for the random matrix ensemble (1.1) reduces to the proof of the limiting relation

$$\lim_{n \to \infty} [\rho_n(\lambda_0)]^{-1} K_n\left(\lambda_0 + \frac{x}{n\rho_n(\lambda_0)}, \lambda_0 + \frac{y}{n\rho_n(\lambda_0)}\right) = \frac{\sin \pi(x-y)}{\pi(x-y)} \tag{1.22}$$

which can be rewritten as

$$\lim_{n \to \infty} [k_n(\lambda_0, \lambda_0)]^{-1} k_n\left(\lambda_0 + \frac{x}{k_n(\lambda_0, \lambda_0)}, \lambda_0 + \frac{y}{k_n(\lambda_0, \lambda_0)}\right) = \frac{\sin \pi(x-y)}{\pi(x-y)} \tag{1.23}$$

and be regarded as a conjecture of purely analytic nature concerning the orthogonal polynomial (1.13). Since for a complete systems of orthonormal functions we have the relation

$$\sum_{j=0}^{\infty} \psi_j^{(n)}(\lambda)\, \psi_j^{(n)}(\mu) = \delta(\lambda - \mu) \tag{1.24}$$

the result (1.23) can be viewed as saying that the fine ("magnified") structure of the δ-function in (1.24) is universal and is given by the r.h.s. of (1.23). The result (1.23) can be readily proven if a precise enough asymptotic formula for the respective orthogonal polynomials is known. Let us consider the simplest ("toy") case of an n-independent weight supported on a finite interval, say the interval $[-1, 1]$. By using classical asymptotic formulas[10] we find that in this case $\rho(\lambda) = (\pi \sqrt{1-\lambda^2})^{-1}$, $|\lambda| \leqslant 1$, and relation (1.9) is valid for any $|\lambda| < 1$. A less trivial case corresponds to the weight (1.12) in which $V(\lambda) = |\lambda|^\alpha/\alpha$ with a positive α. In this case $p_l^{(n)}(\lambda) = n^{1/2\alpha}\pi_l(n^{1/\alpha}\lambda)$, where $\langle \pi_l(x) \rangle_{l=0}^{\infty}$ are orthogonal polynomials associated with the n-independent weight $w(x) = \exp\{-|x|^\alpha/\alpha\}$. The case $\alpha = 2$ corresponds to the Gaussian unitary ensemble and the Hermite polynomials as $\pi_l(x)$. This case was studied in great detail[1] on the basis of the Plancherel–Rotah asymptotic formula[10] describing the semiclassical regime of a quantum oscillator. For the general case $\alpha > 1$ asymptotic formulas were recently obtained in refs. 11 and 12. By using these formulas the limiting density $\rho(\lambda)$ can be found and the relation (1.9) can be checked for $\lambda = 0$.[13] Unfortunately, the asymptotic formulas[11,12] are not precise enough to prove (1.9) for $\lambda \neq 0$. This can be done only for $\alpha = 4, 6$, where more precise asymptotic formulas are known.

3. We mention other works related to the subject of this paper. In ref. 14 a scheme of proof (1.22) for $\lambda = 0$ is proposed. It is based on a formalism developed in studying the so-called double scaling limit of quantum field theory. In the ref. 15 a new asymptotic formula for the orthogonal polynomials $P_l^{(n)}(\lambda)$, $l = n + o(1)$, is proposed, in the case when the support of the density of states $\rho(\lambda)$ is an interval. By using this formula the authors derived (1.22) and, moreover, found a new asymptotic regime for the smoothed correlation function of eigenvalues for $1 \gg \lambda \gg n^{-1}$ ("mesoscopic" scale). These results were improved and developed in a subsequent paper.[16] In ref. 17 the universality conjecture was considered by studying the generating functional of the densities (1.18), which was computed by applying the Laplace method to its Grassman integral representation.

4. We would like to stress that our approach is "local," i.e., it is not sensitive to the form of the support of $\rho(\lambda)$, provided that $\rho(\lambda) > 0$. On the other hand, it is known that if $V(\lambda)$ is a polynomial of degree $2m$, then the support of $\rho(\lambda)$ may consist of several (at most m) intervals. The work in refs. 15 and 16 is based on the asymptotic formulas for the orthogonal polynomials with the weight $e^{-nV(x)}$ that are valid for such $V(\lambda)$, which produce the one-interval support of $\rho(\lambda)$. This is the case if, for instance, $V(\lambda)$ is a convex function (not necessary a polynomial).[9] These papers, while not rigorous, contain essential results and constitute important advances in the problem.

We will prove the Theorem by using the orthogonal polynomial technique, which is rather powerful and widely used in the random matrix theory and its numerous applications. However, since the asymptotic formulas for the general case treated in the Theorem are not known, we combine the orthogonal polynomial technique with certain identities that were introduced in the random matrix theory in the seminal paper of Bessis *et al.*[5]

This paper is organized as follows. In Section 2 we give the proof of the theorem following the main line of the arguments. The important ingredient of our arguments is the pointwise convergence of $\rho_n(\lambda)$ to $\rho(\lambda)$ on the set $\{\lambda: \rho(\lambda) > 0\}$.

Proposition. Under the conditions of the Theorem we have for all λ and n such that $\rho(\lambda) > n^{-1/9}$

$$|\rho_n(\lambda) - \rho(\lambda)| \leqslant C\left(1 + \frac{1}{\rho(\lambda)}\right) n^{-1/4} \tag{1.25}$$

for some positive n-independent constant C.

The Proposition is also proved in Section 2. Auxiliary facts which we need to establish the Theorem and the Proposition are proved in Section 3. We discuss some consequences of our results in Section 4.

2. PROOFS OF THE PROPOSITION AND THE THEOREM

Proof of the Proposition. Consider the Stieltjes transform of the normalized counting measure (1.3)

$$f_n(z) \equiv \int \frac{N_n(d\lambda)}{\lambda - z} = \frac{1}{n} \sum_{l=1}^{n} \frac{1}{\lambda_l - z} \tag{2.1}$$

and denote

$$g_n(z) \equiv E\{f_n(z)\} = \int \frac{\rho_n(\lambda)\, d\lambda}{\lambda - z} \tag{2.2}$$

According to the spectral theorem

$$f_n(z) = \frac{1}{n} \operatorname{Tr} G(z)$$

where $G(z) = (M - z)^{-1}$ is the resolvent of a Hermitian matrix M. By using Lemma 1 (see Section 3) for $F(M) = G_{ik}(z)$ (a matrix element of the resolvent) and $B = B^{(ik)} = \{B_{jm}^{(ik)}\}_{j,m=1}^{n}$, $B_{jm}^{(ik)} = \zeta\delta_{ij}\delta_{km} + \bar{\zeta}\delta_{im}\delta_{kj}$, where $\zeta \in \mathbf{C}$ is a free parameter, it is easy to derive the identity[5]

$$E\{\zeta G_{ii} G_{kk} + \bar{\zeta} G_{ik}^2 + n G_{ik}(\zeta(V'(M))_{ki} + \bar{\zeta}(V'(M))_{ik}) = 0$$

Since ζ is arbitrary, we conclude that

$$E\{G_{ii} G_{kk} + n G_{ik}(V'(M))_{ki}\} = 0$$

Now if we sum this inequality over $i, k = 1,..., n$ and divide the result by n^2, we get

$$E\{f_n^2\} + E\{n^{-1} \operatorname{Tr} V'(M)\, G(z)\} = 0 \tag{2.3}$$

By applying Lemma 3 to $f(\mu) = (\mu - z)^{-1}$, $z = \lambda + i\eta$, $\eta > 0$, we find that

$$E\{f_n^2\} = E^2\{f_n\} + O(n^{-2}\eta^{-4}) \tag{2.4}$$

This bound, (2.1), and (2.2) yield the relation

$$g_n^2(z) + V'(\lambda)\, g_n(z) + Q_n(z) = O(n^{-2}\eta^{-4}) \tag{2.5}$$

where

$$Q_n(z) = \int \frac{V'(\mu) - V'(\lambda)}{\mu - z} \rho_n(\mu)\, d\mu$$

is well defined due to (1.7) and our conditions on $V(\lambda)$ (see the Theorem and Remark 1). To proceed further we use the result (1.8), combining it with condition (1.11b). We obtain

$$Q_n(\lambda + i\eta) = Q(\lambda) + O(\eta^{-1/4} n^{-1/2} \log^{1/2} n) + O(\eta) \tag{2.6}$$

where

$$Q(\lambda) = \int \frac{V'(\mu) - V'(\lambda)}{\mu - \lambda} \rho(\mu)\, d\mu \tag{2.7}$$

Combining (2.5) and (2.6), we find that

$$g_n(\lambda + in^{-2/3}) = -\frac{V'(\lambda)}{2} + \left[\left(\frac{V'(\lambda)}{2} \right)^2 - Q(\lambda) + O(n^{-1/3} \log^{1/2} n) \right]^{1/2} \tag{2.8}$$

Thus on the basis of Lemma 4 we get

$$\pi^{-1} \Im g_n(\lambda + in^{-1/3}) = \rho(\lambda) + O(n^{-1/3} \log^{1/2} n) \frac{1}{\rho(\lambda)} \tag{2.9}$$

On the other hand, it follows from Lemmas 5 and 6 that for λ such that $\rho(\lambda) > n^{-1/9}$

$$|\pi^{-1} \Im g_n(\lambda + in^{-1/3}) - \rho_n(\lambda)| \leqslant C \left(1 + \frac{1}{\rho(\lambda)} \right) n^{-1/4}$$

This bound and (2.9) imply (1.25).

Proof of the Theorem. According to (1.18), the proof of the Theorem reduces to the proof of the limiting relation (1.22) to the reproducing kernel (1.21) of the orthonormal systems (1.14). We use the representation

$$K_n(\lambda, \mu) = Z_n^{-1} \int \prod_{j=2}^{n} d\lambda_j (\lambda - \lambda_j)(\mu - \lambda_j) \prod_{2 \leqslant j < k \leqslant n} (\lambda_j - \lambda_k)^2$$

$$\times \exp\left\{ -\frac{n}{2} V(\lambda) - \frac{n}{2} V(\mu) - \sum_{j=2}^{n} V(\lambda_j) \right\} \tag{2.10}$$

which can be derived from the following identities well known in the RMT[1]:

$$\prod_{1\leqslant j<k\leqslant n}(\lambda_j-\lambda_k)=\left(\prod_{l=1}^{n-1}\gamma_l^{(n)}\right)^{-1}\det\|P_{j-1}^{(n)}(\lambda_k)\|_{j,k=1}^{n}$$

$$Z_n=n!\prod_{l=0}^{n-1}(\gamma_l^{(n)})^{-2}$$

where $\gamma_l^{(n)}$ is the coefficient in front of λ^l in the polynomial $P_l^{(n)}$. If we substitute these identities into the r.h.s. of (2.10), set in the one of the determinant $\lambda_1=\lambda$, in other $\lambda_1=\mu$, and then integrate the result with respect to $\lambda_2,\dots,\lambda_n$, using the orthogonality of polynomials $P_l^{(n)}$, we obtain the l.h.s. of (2.10).

We will consider the function $K_n(\lambda_0,\lambda_0+s/n)$. The general case of the function $K_n(\lambda_0+s/n,\ \lambda_0+t/n)$ can be reduced to $K_n(\lambda_0,\lambda_0+(s-t)/n)$ by using Lemma 7. Let us choose

$$\delta=\frac{\log n}{n}$$

and rewrite (2.10) in the form

$$K_n\left(\lambda_0,\lambda_0+\frac{s}{n}\right)$$
$$=\exp\left\{\frac{n}{2}V(\lambda_0)-\frac{n}{2}V\left(\lambda_0+\frac{s}{n}\right)\right\}\left\langle\prod_{j=2}^{n}\left[\frac{s}{n}\cdot\frac{\chi_\delta(\lambda_0-\lambda_j)}{\lambda_0-\lambda_j}+e^{u(\lambda_j)}\right]\right\rangle \quad (2.11)$$

Here and below the symbol $\langle\dots\rangle$ denotes the operation $E\{\delta(\lambda_0-\lambda_1)\dots\}$, $\chi_\delta(\lambda)$ is the indicator of the interval $|\lambda|\leqslant\delta$, and

$$u(\lambda)=(1-\chi_\delta(\lambda_0-\lambda))\log\left(1+\frac{s}{n(\lambda_0-\lambda)}\right)$$

Rewrite (2.11) as

$$K_n\left(\lambda_0,\lambda_0+\frac{s}{n}\right)$$
$$=T_n(\lambda_0)\left[1+\sum_{l=1}^{n-1}C_{n-1}^l\left(\frac{s}{n}\right)^l\left\langle\prod_{j=2}^{l+1}\frac{\chi_\delta(\lambda_0-\lambda_j)}{\lambda_0-\lambda_j}e^{U_n(\lambda_0)}\right\rangle Z_n^{-1}(\lambda_0)\right] \quad (2.12)$$

where $C_n^l = n!/[l!\,(n-l)!]$ and

$$T_n(\lambda_0) = \exp\left\{-\frac{n}{2}V(\lambda_0) + \frac{n}{2}V\left(\lambda_0 + \frac{s}{n}\right)\right\} Z_n(\lambda_0) \tag{2.13}$$

$$Z_n(\lambda_0) = \langle e^{U_n(\lambda_0)}\rangle \tag{2.14}$$

$$U_n(\lambda_0) = \sum_{j=2}^{n} u(\lambda_j) \tag{2.15}$$

Introduce the probability density [cf. (1.16)]

$$p_{tn}(\lambda_2,\dots,\lambda_n) = Z_{tn}^{-1}(\lambda_0)\prod_{j=2}^{n}(\lambda_0-\lambda_j)^2 \prod_{2\leqslant j<k\leqslant n}(\lambda_j-\lambda_k)^2$$

$$\times \exp\left\{-n\sum_{j=2}^{n}V(\lambda_j) + t\sum_{j=2}^{n}u(\lambda_j)\right\} \tag{2.16}$$

where $Z_{tn}^{-1}(\lambda_0)$ is the normalization factor, and the respective marginal densities

$$p_{tl}^{(n)}(\lambda_2,\dots,\lambda_{l+1}) = \int p_{tn}(\lambda_2,\dots,\lambda_n)\,d\lambda_{l+2}\dots d\lambda_n \tag{2.17}$$

In particular, for $t=0$

$$p_{0l}^{(n)}(\lambda_2,\dots,\lambda_{l+1}) = \rho_n^{-1}(\lambda_0)\,p_{l+1}^{(n)}(\lambda_0,\lambda_2,\dots,\lambda_{l+1}) \tag{2.18}$$

This allows us to rewrite (2.12) as follows:

$$K_n\left(\lambda_0,\lambda_0+\frac{s}{n}\right) = T_n(\lambda_0)\left[1+\sum_{l=1}^{n-1}C_{n-1}^l\left(\frac{s}{n}\right)^l\right.$$

$$\left.\times\int\prod_{j=2}^{l+1}\frac{\chi_\delta(\lambda_0-\lambda_j)}{\lambda_0-\lambda_j}\,p_{1n}(\lambda_2,\dots,\lambda_{l+1})\,d\lambda_2\dots d\lambda_{l+1}\right] \tag{2.19}$$

Introduce

$$R_{tn}(\lambda,\mu) = \frac{1}{n-1}\sum_{k=0}^{n-2}\psi_{tk}(\lambda)\,\psi_{tk}(\mu) \tag{2.20}$$

where

$$\psi_{tk}(\lambda) = (\lambda-\lambda_0)\exp\left\{-\frac{n}{2}V(\lambda)+\frac{t}{2}u(\lambda)\right\}P_{tl}(\lambda)$$

and $\{P_{tk}(\lambda)\}_{k=0}^{\infty}$ are polynomials that are orthogonal with respect to the weight $(\lambda-\lambda_0)^2\exp\{-nV(\lambda)+tu(\lambda)\}$.

$$\int P_{tl}(\lambda)\,P_{tm}(\lambda)(\lambda-\lambda_0)^2\exp\{-nV(\lambda)+tu(\lambda)\}\,d\lambda=\delta_{lm}$$

Then [cf. (1.18)]

$$p_{tl}^{(n)}(\lambda_1,\dots,\lambda_{l+1})=\frac{(n-1)^l}{(n-1)\dots(n-l-1)}\det\|R_{tn}(\lambda_{j+1},\lambda_{k+1})\|_{j,k=1}^{l}$$

and after the change of variables $x_j=n(\lambda_{j+1}-\lambda_0)$, we can write (2.19) in the form

$$K_n\left(\lambda_0,\lambda_0+\frac{s}{n}\right)=T_n(\lambda_0)\left[1+\sum_{l=1}^{n-1}\frac{s^l(n-1)^l}{n^l l!}\right.$$
$$\left.\times\int_{-n\delta}^{n\delta}\prod_{j=1}^{l}\frac{dx_j}{x_j}\det\left\|R_{1n}\left(\lambda_0+\frac{x_j}{n},\lambda_0+\frac{x_k}{n}\right)\right\|_{j,k=1}^{l}\right]\quad(2.21)$$

We will prove that

$$K_n\left(\lambda_0,\lambda_0+\frac{s}{n}\right)=T_n(\lambda_0)\left[1+\sum_{l=1}^{n-1}\frac{s^l}{l!}\int_{-n\delta}^{n\delta}\prod_{j=1}^{l}\frac{dx_j}{x_j}\right.$$
$$\left.\times\det\left\|R_{0n}\left(\lambda_0+\frac{x_j}{n},\lambda_0+\frac{x_k}{n}\right)\right\|_{j,k=1}^{l}+O\left(\frac{1}{n\delta}\right)\right]\quad(2.22)$$

To this end we use Lemma 9. Therefore we have to check conditions (3.41)–(3.45) of the lemma for $A=R_{0n}$ and $B=R_{1n}$. Inequality (3.41) follows from (2.18) and Lemma 8, inequality (3.43) follows from (2.18) and Lemma 7, and inequality (3.42) follows from the representation (2.20). To check (3.44) and (3.45), consider the derivative $R'_{tn}(\lambda_0+x/n,\ \lambda_0+y/n)$ of (2.20) with respect to t. By using arguments similar to those in the proof of Lemma 5, we obtain

$$R'_{tn}\left(\lambda_0+\frac{x}{n},\lambda_0+\frac{y}{n}\right)$$
$$=\frac{1}{2}\left[u\left(\lambda_0+\frac{x}{n}\right)+u\left(\lambda_0+\frac{y}{n}\right)\right]R_{tn}\left(\lambda_0+\frac{x}{n},\lambda_0+\frac{y}{n}\right)$$
$$-(n-2)\int R_{tn}\left(\lambda_0+\frac{x}{n},\mu\right)R_{tn}\left(\lambda_0+\frac{y}{n},\mu\right)u(\mu)\,d\mu\quad(2.23)$$

If $|x| < n\delta$ and $|y| < n\delta$, then the first term in the r.h.s. of (2.23) is zero. The second term can be estimated by using the Schwarz inequality and the analogue of (3.8) for (2.20),

$$\begin{aligned}
&\left|(n-2)\int R_{tn}\left(\lambda_0+\frac{x}{n},\mu\right)R_{tn}\left(\lambda_0+\frac{y}{n},\mu\right)u(\mu)\,d\mu\right| \\
&\quad\leqslant\left|(n-2)\int R_{tn}\left(\lambda_0+\frac{x}{n},\mu\right)^2|u(\mu)|\,d\mu\right|^{1/2} \\
&\qquad\times\left|(n-2)\int R_{tn}\left(\lambda_0+\frac{y}{n},\mu\right)^2|u(\mu)|\,d\mu\right|^{1/2} \\
&\quad\leqslant\max_{\lambda}|u(\lambda)|\cdot\left[R_{tn}\left(\lambda_0+\frac{x}{n},\lambda_0+\frac{x}{n}\right)\cdot R_{tn}\left(\lambda_0+\frac{y}{n},\lambda_0+\frac{y}{n}\right)\right]^{1/2}
\end{aligned}\tag{2.24}$$

Hence

$$\begin{aligned}
&\left|R'_{tn}\left(\lambda_0+\frac{x}{n},\lambda_0+\frac{y}{n}\right)\right| \\
&\qquad\leqslant\frac{C}{n\delta}\left[R_{tn}\left(\lambda_0+\frac{x}{n},\lambda_0+\frac{x}{n}\right)R_{tn}\left(\lambda_0+\frac{y}{n},\lambda_0+\frac{y}{n}\right)\right]^{1/2}
\end{aligned}\tag{2.25}$$

In addition,

$$\begin{aligned}
&\max_t R_{tn}\left(\lambda_0+\frac{x}{n},\lambda_0+\frac{x}{n}\right) \\
&\quad=R_{t^*n}\left(\lambda_0+\frac{x}{n},\lambda_0+\frac{x}{n}\right) \\
&\quad=R_{0n}\left(\lambda_0+\frac{x}{n},\lambda_0+\frac{x}{n}\right)+\int_0^{t^*}d\tau\,R'_{\tau n}\left(\lambda_0+\frac{x}{n},\lambda_0+\frac{x}{n}\right) \\
&\quad\leqslant R_{0n}\left(\lambda_0+\frac{x}{n},\lambda_0+\frac{x}{n}\right)+\frac{C}{n\delta}R_{t^*n}\left(\lambda_0+\frac{x}{n},\lambda_0+\frac{x}{n}\right)
\end{aligned}\tag{2.26}$$

Thus it follows from (2.26) that for all x and t [cf. (3.44)]

$$R_{tn}\left(\lambda_0+\frac{x}{n},\lambda_0+\frac{x}{n}\right)\leqslant CR_{0n}\left(\lambda_0+\frac{x}{n},\lambda_0+\frac{x}{n}\right)\tag{2.27}$$

Combining (2.25) and (2.27), we obtain [cf. (3.44)]

$$\left|R_{1n}\left(\lambda_0+\frac{x}{n},\lambda_0+\frac{y}{n}\right)-R_{0n}\left(\lambda_0+\frac{x}{n},\lambda_0+\frac{y}{n}\right)\right|$$
$$\leqslant\frac{C}{n\delta}R_{0n}^{1/2}\left(\lambda_0+\frac{x}{n},\lambda_0+\frac{x}{n}\right)R_{0n}^{1/2}\left(\lambda_0+\frac{y}{n},\lambda_0+\frac{y}{n}\right)\tag{2.28}$$

Inequality (2.28), identity (2.18), and Lemma 8 guarantee condition (3.44) of Lemma 9. Condition (3.45) can be proved by similar arguments. Thus we can apply Lemma 9 to the expression in the r.h.s. of (2.21) and obtain (2.22).

By using the analogue of the representation (2.10) for $R_{0n}(\lambda,\mu)$ we get

$$R_{0n}(\lambda,\mu)=K_n(\lambda,\mu)-\frac{K_n(\lambda_0,\lambda)\,K_n(\lambda_0,\mu)}{K_n(\lambda_0,\lambda_0)}\tag{2.29}$$

We will use this representation to prove that we can replace the function $R_{0n}(\lambda,\mu)$ in the r.h.s. of (2.22) by

$$R^*(x_j,x_k)=K_n\left(\lambda_0,\lambda_0+\frac{x_k-x_j}{n}\right)-\frac{K_n(\lambda_0,\lambda_0-x_j/n)\,K_n(\lambda_0,\lambda_0+x_k/n)}{K_n(\lambda_0,\lambda_0)}\tag{2.30}$$

We use again Lemma 9 for $A=R_{0n}$, $B=R^*$. As explained above, conditions (3.41)–(3.43) hold for this A and thus we have to check (3.44) and (3.45). Since, according to (2.29) and (2.30),

$$|R^*(x,y)-R_{0n}(x,y)|$$
$$\leqslant\left|K_n\left(\lambda_0,\lambda_0+\frac{y-x}{n}\right)-K_n\left(\lambda_0+\frac{x}{n},\lambda_0+\frac{y}{n}\right)\right|$$
$$+\left|\frac{K_n(\lambda_0,\lambda_0+y/n)}{K_n(\lambda_0,\lambda_0)}\right|\cdot\left|K_n\left(\lambda_0,\lambda_0+\frac{x}{n}\right)-K_n\left(\lambda_0,\lambda_0-\frac{x}{n}\right)\right|$$

it suffices to check that uniformly in $|y|\leqslant n\delta$ and $n\to\infty$

$$\left|K_n\left(\lambda_0+\frac{x}{n},\lambda_0+\frac{y}{n}\right)-K_n\left(\lambda_0,\lambda_0+\frac{y-x}{n}\right)\right|^2\leqslant\frac{Cx^2}{n^{1/4}},\qquad |x|\leqslant 1\tag{2.31}$$

and

$$\int_{-n\delta}^{n\delta}\left|K_n\left(\lambda_0+\frac{x}{n},\lambda_0+\frac{y}{n}\right)-K_n\left(\lambda_0,\lambda_0+\frac{x-y}{n}\right)\right|^2 dx\leqslant\frac{C(n\delta)^3}{n^{1/4}} \tag{2.32}$$

Estimate (2.31) follows from Lemma 7, because $|x|$, $|y|\leqslant n\delta=\log n$. Estimate (2.32) can be obtained if we integrate (2.31) with respect to x. Thus we have proved that

$$K_n\left(\lambda_0,\lambda_0+\frac{s}{n}\right)=T_n(\lambda_0)\left[1+\sum_{l=1}^{n-1}\frac{s^l}{l!}\int_{-n\delta}^{n\delta}\prod_{j=1}^{l}\frac{dx_j}{x_j}\right.$$
$$\left.\times\det\|R^*(x_j,x_k)\|_{j,k=1}^{l}+O\left(\frac{1}{n\delta}\right)\right] \tag{2.33}$$

The next step is to prove that we can replace the integral over the interval $(-n\delta,n\delta)$ in the r.h.s. of (2.33) by the integral over the whole axis $\mathbf{R}$. To this end let us notice first that since $R^*(x,x)=R^*(-x,-x)$

$$\int_{-n\delta}^{n\delta}\prod_{j=1}^{l}\frac{dx_j}{x_j}\det\|R^*(x_j,x_k)\|_{j,k=1}^{l}$$
$$=\int_{-n\delta}^{n\delta}\prod_{j=1}^{l}\frac{dx_j}{x_j}\det\|R^*(x_j,x_k)(1-\delta_{jk})\|_{j,k=1}^{l} \tag{2.34}$$

In addition,

$$\Delta_l\equiv\left|\int\sum_{j=1}^{l}\frac{dx_j}{x_j}\det\|R^*(x_j,x_k)(1-\delta_{jk})\|_{j,k=1}^{l}\right.$$
$$\left.-\int_{-n\delta}^{n\delta}\prod_{j=1}^{l}\frac{dx_j}{x_j}\det\|R^*(x_j,x_k)(1-\delta_{jk})\|_{j,k=1}^{l}\right|$$
$$\leqslant\sum_{p=1}^{l}C_l^p\int\prod_{j=1}^{p}\frac{(1-\chi_{n\delta}(x_j))\,dx_j}{|x_j|}$$
$$\times\prod_{j=p+1}^{l}\frac{\chi_{n\delta}(x_j)\,dx_j}{|x_j|}\left|\det\|R^*(x_j,x_k)(1-\delta_{jk})\|_{j,k=1}^{l}\right|$$

$$
\begin{aligned}
&\leqslant \sum_{p=1}^{l} C_l^p \sum_{m=0}^{l-p} C_{l-p}^m \int \prod_{j=1}^{p} \frac{(1-\chi_{n\delta}(x_j))\,dx_j}{|x_j|} \\
&\quad \times \prod_{i=p+1}^{p+m} \frac{(1-\chi_1(x_i))\,dx_i}{|x_j|} \\
&\quad \times \prod_{k=p+m+1}^{l} \frac{\chi_1(x_k)\,dx_k}{|x_k|} \left|\det \|R^*(x_j, x_k)(1-\delta_{jk})\|_{j,k=1}^{l}\right| \\
&\leqslant \sum_{p=1}^{l} C_l^p \sum_{m=0}^{l-p} C_{l-p}^m \left\{ \int \prod_{j=1}^{p} \frac{(1-\chi_{n\delta}(x_j))\,dx_j}{|x_j|^{5/4}} \right. \\
&\quad \left. \times \prod_{i=p+1}^{p+m} \frac{(1-\chi_1(x_i))\,dx_i}{|x_j|^{5/4}} \prod_{k=p+m+1}^{l} dx_k\,\chi_1(x_k) \right\}^{1/2} \\
&\quad \times \left\{ \int \prod_{j=1}^{p} \frac{(1-\chi_{n\delta}(x_j))\,dx_j}{|x_j|^{3/4}} \prod_{i=p+1}^{p+m} \frac{(1-\chi_1(x_i))\,dx_i}{|x_j|^{3/4}} \right. \\
&\quad \left. \times \prod_{k=p+m+1}^{l} \frac{\chi_1(x_k)\,dx_k}{|x_k|^2} \left|\det \|R^*(x_j, x_k)(1-\delta_{jk})\|_{j,k=1}^{l}\right|^2 \right\}^{1/2}
\end{aligned}
$$

The first factor in the r.h.s. of the last inequality can be estimated by $(n\delta)^{-p/4}C^{l-p}$. To estimate the second one we repeat almost literally the arguments of Lemma 9. We obtain

$$
\Delta_l \leqslant l^{(l+2)/2} C^l (n\delta)^{-1/4}
$$

Therefore

$$
\begin{aligned}
K_n\left(\lambda_0, \lambda_0 + \frac{s}{n}\right) = T_n(\lambda_0) \Bigg[1 + \sum_{l=1}^{n-1} \frac{s^l}{l!} \int \prod_{j=1}^{l} \frac{dx_j}{x_j} \\
\times \det \|R^*(x_j, x_k)\|_{j,k=1}^{l} + O\left(\frac{1}{(n\delta)^{1/4}}\right) \Bigg]
\end{aligned} \tag{2.35}
$$

Now, by using the formula

$$
\det \|a_{jk}\|_{j,k=0}^{l} = a_{00} \det \left\| a_{jk} - \frac{a_{j0}a_{0k}}{a_{00}} \right\|_{j,k=1}^{l}
$$

we obtain from (2.35)

$$K_n\left(\lambda_0, \lambda_0+\frac{s}{n}\right)=T_n(\lambda_0)\left[1+\sum_{l=1}^{n-1}\frac{s^l}{l!\,\rho_n(\lambda_0)}\int\prod_{j=1}^{l}\frac{dx_j}{x_j}\right.$$
$$\left.\times\det\|S_n(x_j-x_k)\|_{j,k=0}^{l}+O\left(\frac{1}{(n\delta)^{1/4}}\right)\right] \tag{2.36}$$

where $x_0=0$ and $S_n(x)=K_n(\lambda_0, \lambda_0+x/n)$. The integral in the r.h.s. of (2.36) can be computed by using the Fourier integral technique. This is done in Lemma 11 of Section 3. According to that lemma,

$$K_n\left(\lambda_0, \lambda_0+\frac{s}{n}\right)=T_n(\lambda_0)\left[\frac{\sin\pi\rho_n(\lambda_0)\,s}{\pi\rho_n(\lambda_0)\,s}+O\left(\frac{1}{(n\delta)^{1/4}}\right)\right] \tag{2.37}$$

Comparing this expression with (1.22), we see that to finish the proof of the Theorem, we have to establish the relation

$$\lim_{n\to\infty} T_n(\lambda_0)=\rho(\lambda_0)$$

This relation follows from the Proposition and Lemma 10 of Section 3. The Theorem is proved.

3. AUXILIARY RESULTS

In this section we prove a number of facts that we used in the proofs of the Theorem and the Proposition in Section 2.

Lemma 1. Let $F(t)$, $t\in\mathbf{R}$, be a continuously differentiable and polynomially bounded function, and let B be an arbitrary Hermitian matrix. Then

$$E\{F'_B(M)\}-nE\{F(M)\operatorname{Tr}V'(M)B\}=0 \tag{3.1}$$

where $F'_B(M)=\lim_{\varepsilon\to\infty}\varepsilon^{-1}[F(M+\varepsilon B)-F(M)]$.

Proof. We obtain the lemma by differentiating with respect to t the identity

$$\int\exp\{-n\operatorname{Tr}V(M+tB)\}\,F(M+tB)\,dM=\int\exp\{-n\operatorname{Tr}V(M)\}\,F(M)\,dM$$

which follows from the invariance of the measure dM with respect to shift $M\to M+B$ by an arbitrary Hermitian matrix B.

Remark. This lemma was in fact proved by Bessis *et al.*[5]

Lemma 2. Let $K_n(\lambda, \mu)$ be defined by (1.21). Then

$$\int (\lambda-\mu)^2 K_n^2(\lambda, \mu)\, d\lambda\, d\mu \leqslant \frac{C}{n^2} \tag{3.2}$$

and for $\alpha = 1, 2$

$$\left|\int (\lambda-\mu)^\alpha K_n^2(\lambda, \mu)\, d\mu\right| \leqslant \frac{C}{n^2} \{(\psi_{n-1}^{(n)}(\lambda))^2 + (\psi_n^{(n)}(\lambda))^2\} \tag{3.3}$$

Proof. It follows from the orthogonality relations (1.15) that for $j = 0, 1, 2,...$

$$r_j P_{j+1}(\lambda) + r_{j-1} P_{j-1}(\lambda) = \lambda P_j(\lambda) \qquad (r_{-1} = 0) \tag{3.4}$$

where

$$r_j = \int \lambda P_j(\lambda)\, P_{j+1}(\lambda)\, e^{-nV(\lambda)}\, d\lambda \tag{3.5}$$

and we omit the superscript n to simplify the notation. Denote by $J = \{J_{jk}\}_{j,k=1}^{\infty}$ the Jacobi matrix defined by (3.4):

$$J_{jk} = r_j \delta_{j+1,k} + r_{j-1} \delta_{j-1,k} \tag{3.6}$$

Then for any nonnegative integer p

$$(J^p)_{jk} = \int \lambda^p \psi_j^{(n)}(\lambda)\, \psi_k^{(n)}(\lambda)(\lambda)\, d\lambda \tag{3.7}$$

By using the identity

$$\int K_n^2(\lambda, \mu)\, d\mu = \frac{1}{n} K_n(\lambda, \lambda) \tag{3.8}$$

and (3.7) for $p = 1, 2$, we find that the l.h.s. of (3.2) is

$$2\left(\sum_{j=0}^{n-1} (J^2)_{jj} - \sum_{j,k=0}^{n-1} J_{jk}^2\right) \tag{3.9}$$

This relation and (3.6) yield

$$n^2 \int (\lambda-\mu)^2 K_n^2(\lambda, \mu)\, d\lambda\, d\mu = 2r_{n-1}^2 \tag{3.10}$$

Using (1.7) and (1.19)–(1.21), we obtain that for some n-independent a, L_1, $L>0$

$$[\psi_l(\lambda)]^2 \leqslant n\rho_n(\lambda) \leqslant n \exp\{-an[V(\lambda) - \max_{|\mu| \leqslant L_1} V(\mu)]\}, \qquad |\lambda| \geqslant L \tag{3.11}$$

and then (3.5) implies the bound

$$|r_l| \leqslant C \tag{3.12}$$

for some C. This bound and (3.10) imply (3.2). Similar arguments and Eq. (3.4) yield

$$n^2 \int (\lambda-\mu)\, K_n^2(\lambda, \mu)\, d\mu = \psi_{n-1}(\lambda)\, \psi_n(\lambda)\, r_{n-1}$$

Now (3.3) follows from this identity and (3.12). The case $\alpha=2$ in the l.h.s. of (3.3) can be proved analogously. Lemma 2 is proved.

Lemma 3. Let $f(\mu)$, $\mu \in \mathbf{R}$, be a bounded and Holder continuous function,

$$|f(\lambda) - f(\mu)| \leqslant C\, |\lambda-\mu|^\alpha \tag{3.13}$$

for some $C>0$ and $0<\alpha \leqslant 1$, and

$$f_n = \frac{1}{n} \sum_{j=1}^{n} f(\lambda_j^{(n)})$$

where $\{\lambda_j^{(n)}\}_{j=1}^n$ are eigenvalues of a random matrix. Then

$$D\{f_n\} \equiv E\{|f_n - E\{f_n\}|^2\} \leqslant C_1 n^{-1-\alpha} \tag{3.14}$$

Proof. By using (1.18) and (1.19), we can write (3.14) as

$$D\{f_n\} = \tfrac{1}{2} \int |f(\lambda) - f(\mu)|^2\, K_n^2(\lambda, \mu)\, d\lambda\, d\mu$$

This representation, (3.13), the Holder inequality, and the relation

$$\int K_n^2(\lambda, \mu)\, d\lambda\, d\mu = n^{-1} \tag{3.15}$$

yield the bound

$$D\{f_n\} \leqslant \frac{C^2}{2}\left[\int |\lambda-\mu|^2 K_n^2(\lambda, \mu)\, d\lambda\, d\mu\right]^{\alpha} [n^{-1}]^{1-\alpha}$$

which implies (3.14) in view of Lemma 2. Lemma 3 is proved.

Lemma 4. Assume that λ is a point of the spectral axis at which $\rho(\lambda) > 0$. Then

$$\rho(\lambda) = \frac{1}{\pi}\{Q(\lambda) - [V'(\lambda)]^2/4\}^{1/2} \tag{3.16}$$

where $Q(\lambda)$ is defined by (2.7).

Proof. According to (1.8), $\rho_n(\lambda)$ converges weakly to $\rho(\lambda)$. This result allows us to perform the limiting transition in (2.5) and to obtain for nonreal z's the relation

$$g^2(z) + V'(\lambda)\, g(z) + Q(z) = 0 \tag{3.17}$$

where $g(z)$ is the Stieltjes transform of the limiting density $\rho(\lambda)$. Definition (2.7) and condition (1.11b) of the Theorem imply that $Q(\lambda + i0)$ is a real-valued, bounded function with a bounded derivative. Then by general principles

$$\rho(\lambda) = \frac{1}{\pi} \Im g(\lambda + i0) \tag{3.18}$$

is also bounded. Computing the real and the imaginary parts of (3.17) rewritten as

$$g = -\frac{Q}{V' + g} \tag{3.19}$$

we find (3.16). Lemma 4 is proved.

Lemma 5. Under the conditions of the Theorem

$$\sup_{n,\lambda} \rho_n(\lambda) \leqslant C \tag{3.20}$$

and

$$\left|\frac{d\rho_n(\lambda)}{d\lambda}\right| \leqslant C_1(\psi_{n-1}^2(\lambda)+\psi_n^2(\lambda))+C_2 \tag{3.21}$$

Proof. We start from a simple identity

$$\frac{d\rho_n(\lambda)}{d\lambda}=\left.\frac{d\rho_n(\lambda+t)}{dt}\right|_{t=0}$$

Performing in the integral (1.2) the change of variables $\lambda_i - t = \mu_i$, $i=2,\dots,n$, we rewrite $\rho_n(\lambda+t)$ as follows:

$$\rho_n(\lambda+t)=Z_n^{-1}\int \exp\left\{-nV(\lambda)-n\sum_{j=2}^{n} V(t+\mu_j)\right\}$$
$$\times\prod_{j=2}^{n}(\lambda-\mu_j)^2\prod_{i>j\geqslant 2}^{n}(\mu_i+\mu_j)^2\,d\mu_j$$

Thus, after differentiating with respect to t, we get, for $t=0$,

$$\begin{aligned}\frac{d\rho_n(\lambda)}{d\lambda} &= -nV'(\lambda)\,\rho_n(\lambda)-n(n-1)\int V'(\lambda_2)\,p_n(\lambda,\lambda_2,\dots,\lambda_n)\,d\lambda_2\dots d\lambda_n\\ &= -nV'(\lambda)\,K_n(\lambda,\lambda)\\ &\quad -n^2\int V'(\lambda_2)[K_n(\lambda,\lambda)\,K_n(\lambda_2,\lambda_2)-K_n^2(\lambda,\lambda_2)]\,d\lambda_2 \end{aligned} \tag{3.22}$$

The identity (3.1) for $F(M)=1$ and $B=1$ yields

$$E\{\mathrm{Tr}\ V'(M)\}=n\int V'(\lambda)\,K_n(\lambda,\lambda)\,d\lambda=0$$

Hence by (3.22)

$$\rho_n'(\lambda)=n^2\int[V'(\mu)-V'(\lambda)]\,K_n^2(\lambda,\mu)\,d\mu \tag{3.23}$$

Now we split this integral into two parts corresponding to the intervals $|\mu|>L$ and $|\mu|\leqslant L$, where L is defined by (1.7). The former integral is bounded because of the inequality $K_n^2(\lambda,\mu)\leqslant K_n(\lambda,\lambda)\,K_n(\mu,\mu)$ and the bound (1.7) for $K_n(\lambda,\lambda)=\rho_n(\lambda)$. In the latter we write

$$V'(\mu)-V'(\lambda)=(\mu-\lambda)\ V''(\lambda)+\frac{(\mu-\lambda)^2}{2}V'''(\xi)$$

for some ξ depending on λ and μ and use Lemma 2 and condition (1.11b) of the Theorem. Combining the bounds for these two integrals, we obtain (3.21). To obtain (3.20), we have to use (3.23) and (1.13). Lemma 5 is proved.

Lemma 6. Take $\varepsilon=O(n^{-1/4})$. Then for any λ and n such that $\rho(\lambda)>n^{-1/9}$ we have

$$\int_{|\mu-\lambda|\leqslant\varepsilon}(\psi_{n-1}^2(\mu)+\psi_n^2(\mu))\,d\mu\leqslant C_1\left(1+\frac{1}{\rho(\lambda)}\right)n^{-1/4} \tag{3.24}$$

$$\frac{1}{n}(\psi_{n-1}^2(\mu)+\psi_n^2(\mu))\leqslant C_2\left(1+\frac{1}{\rho(\lambda)}\right)n^{-1/8},\qquad |\mu-\lambda|\leqslant\varepsilon \tag{3.25}$$

Proof. Let us introduce the density

$$p_n^-(\lambda_1,\dots,\lambda_{n-1})=\frac{1}{Z_n^-}\exp\left\{-n\sum_{j=1}^{n-1}V(\lambda_j)\right\}\prod_{1\leqslant j<k\leqslant n-1}(\lambda_j-\lambda_k)^2 \tag{3.26}$$

The difference of this density from density (1.2) written for $n-1$ variables $\lambda_1,\dots,\lambda_{n-1}$ is that in the former we have the factor n in the exponent, while in the latter we have $n-1$. Set

$$\rho_n^-(\lambda)=\frac{n-1}{n}\int p_n^-(\lambda,\lambda_2,\dots,\lambda_{n-1})\,d\lambda_2\dots d\lambda_{n-1}=\frac{1}{n}\sum_{j=0}^{n-2}[\psi_j(\lambda)]^2 \tag{3.27}$$

Then

$$\psi_{n-1}^2(\lambda)=n[\rho_n(\lambda)-\rho_n^-(\lambda)] \tag{3.28}$$

Furthermore, by using the analogue of identity (3.1) for the density p_n^- and arguments similar to those proving (2.5), we obtain the relation

$$[g_n^-(z)]^2+\int\frac{V'(\mu)\,\rho_n^-(\mu)}{\mu-z}\,d\mu=O\left(\frac{1}{n^2\eta^4}\right) \tag{3.29}$$

for the Stieltjes transform $g_n^-(z)$ of $\rho_n^-(\mu)$ and $z=\lambda+i\eta$, $\eta>0$. Denote

$$\Delta_n(z)\equiv n(g_n(z)-g_n^-(z))\equiv\int\frac{\psi_{n-1}^2(\mu)}{\mu-z}\,d\mu \tag{3.30}$$

subtract (3.29) from (2.5), and multiply the result by n. We obtain

$$\Delta_n(z)(g_n(z)+g_n^-(z))+\int \frac{V'(\mu)\,\psi_{n-1}^2(\mu)}{\mu-z}\,d\mu = O\left(\frac{1}{n\eta^4}\right)$$

For $z=\lambda+in^{-1/4}$ this relation takes the form

$$\Delta_n(z)(g_n(z)+g_n^-(z)-V'(\lambda))=\int \frac{(V'(\lambda)-V'(\mu))\,\psi_{n-1}^2(\mu)}{\mu-z}\,d\mu+O(1)$$

Then relations (2.8), (3.16), and (3.18) imply that

$$\Im\Delta_n(\lambda+in^{-1/4})\leqslant C_3\left(1+\frac{1}{\rho(\lambda)}\right)$$

Using definition (3.30), we obtain for $z=\lambda+i\varepsilon$ with $\varepsilon=n^{-1/4}$

$$\int_{|\mu-\lambda|\leqslant\varepsilon}\psi_{n-1}^2(\mu)\,d\mu\leqslant 2\varepsilon^2\int\frac{\psi_{n-1}^2(\mu)}{(\mu-\lambda)^2+\varepsilon^2}\,d\mu\leqslant C_4\left(1+\frac{1}{\rho(\lambda)}\right)\varepsilon \quad (3.31)$$

Now we derive (3.25) from (3.31). Set

$$\Psi_{n-1}=\psi_{n-1}^2(\mu^*)=\max_{|\mu-\lambda|\leqslant\varepsilon}\{\psi_{n-1}^2(\mu)\}$$

$$\mu_1=\sup\{\mu\colon \mu\in(\lambda-\varepsilon,\mu^*),\ \psi_{n-1}^2(\mu)\leqslant\Psi_{n-1}/2\}$$

Since $|\mu_1-\lambda|\leqslant\varepsilon$, we have from (3.31)

$$\frac{\Psi_{n-1}}{2}(\mu^*-\mu_1)\leqslant\int_{\mu_1}^{\mu_*}\psi_{n-1}^2(\mu)\,d\mu\leqslant C_4\left(1+\frac{1}{\rho(\lambda)}\right)n^{-1/4} \quad (3.32)$$

On the other hand,

$$\left(\frac{\Psi_{n-1}^{1/2}-(\frac{1}{2}\Psi_{n-1})^{1/2}}{\mu^*-\mu_1}\right)^2(\mu^*-\mu_1)\leqslant\int_{\mu_1}^{\mu^*}(\psi_{n-1}')^2(\mu)\,d\mu\leqslant\int(\psi_{n-1}')^2(\mu)\,d\mu$$

and since

$$\int(\psi_{n-1}')^2(\mu)\,d\mu=\int\frac{n^2}{4}(V'(\mu))^2(\psi_{n-1})^2(\mu)\,d\mu\leqslant C_5n^2$$

we obtain

$$\frac{\Psi_{n-1}}{\mu^*-\mu_1} \leqslant C_5 n^2 \tag{3.33}$$

Now if we multiply (3.32) by (3.33), we get (3.25) for ψ_{n-1}.

To prove the analogous bounds for ψ_n^2, we have to repeat the above arguments for the density [cf. (3.26)]

$$p_n^+(\lambda_1,..., \lambda_{n+1}) = \frac{1}{Z_n^+} \exp\left\{-n \sum_{j=1}^{n+1} V(\lambda_j)\right\} \prod_{1 \leqslant j<k \leqslant n+1} (\lambda_j - \lambda_k)^2$$

and for

$$\rho_n^+(\lambda) = \frac{n+1}{n} \int p_n^+(\lambda, \lambda_2,..., \lambda_{n+1})\, d\lambda_2 \dots d\lambda_{n+1} = \frac{1}{n} \sum_{j=0}^{n} [\psi_j(\lambda)]^2$$

so that $\psi_n^2(\lambda) = n(\rho_n^+(\lambda) - \rho_n(\lambda))$ [cf. (3.28)]. Lemma 6 is proved.

Lemma 7. If $\rho(\lambda_0) \neq 0$, then

$$\begin{aligned} &\left| K_n\left(\lambda_0 + \frac{x}{n}, \lambda_0 + \frac{y}{n}\right) - K_n\left(\lambda_0, \lambda_0 + \frac{y-x}{n}\right)\right| \\ &\qquad \leqslant C\,|x| \left(\frac{1}{n^{1/8}} + \frac{|x-y|^2}{n^2} + e^{-nad/2}\right) \end{aligned} \tag{3.34}$$

Proof. Repeating almost literally the derivation of (3.22), we get

$$\begin{aligned} &\frac{d}{dt} K_n\left(\lambda_0 + \frac{x-tx}{n}, \lambda_0 + \frac{y-tx}{n}\right) \\ &\quad = xn \int K_n\left(\lambda_0 + \frac{x-tx}{n}, \lambda\right) K_n\left(\lambda_0 + \frac{y-tx}{n}, \lambda\right) \\ &\qquad \times \left[V'(\lambda) - \frac{1}{2} V'\left(\lambda_0 + \frac{x-tx}{n}\right) - \frac{1}{2} V'\left(\lambda_0 + \frac{y-tx}{n}\right)\right] d\lambda \end{aligned}$$

To estimate the r.h.s. of this relation, we split this integral into two parts corresponding to the intervals $|\lambda| > L$ and $|\lambda| \leqslant L$, where L is defined by (1.7). The former integral is bounded by $C \exp\{-nad/2\}$ because of the inequality $K_n^2(\lambda, \mu) \leqslant K_n(\lambda, \lambda)\, K_n(\mu, \mu)$ and the bound (1.7) for $K_n(\lambda, \lambda) = \rho_n(\lambda)$. In the latter integral we write

$$V'(\lambda)-\frac{1}{2}V'\left(\lambda_0+\frac{x-tx}{n}\right)-\frac{1}{2}V'\left(\lambda_0+\frac{y-tx}{n}\right)$$
$$=\frac{1}{2}(\lambda-\lambda_x)\,V''(\lambda_x)+\frac{1}{2}(\lambda-\lambda_y)\,V''(\lambda_y)$$
$$+\,O((\lambda-\lambda_x)^2+(\lambda-\lambda_y)^2)$$
$$=\frac{1}{2}(\lambda-\lambda_x)\,V''(\lambda_x)+\frac{1}{2}(\lambda-\lambda_y)\,V''(\lambda_y)$$
$$+\,O\left((\lambda-\lambda_x)(\lambda-\lambda_y)+\frac{|x-y|^2}{n^2}\right)$$

where

$$\lambda_x=\lambda_0+\frac{x-tx}{n},\qquad \lambda_y=\lambda_0+\frac{y-tx}{n}$$

According to (3.7),

$$n\int K_n(\lambda_x,\lambda)\,K_n(\lambda_y,\lambda)(\lambda-\lambda_{x,y})\,d\lambda=\frac{r_{n-1}}{n}\,\psi_n^{(n)}(\lambda_x)\,\psi_n^{(n)}(\lambda_y)$$

In addition, by the Schwartz inequality

$$n\left|\int K_n(\lambda_x,\lambda)\,K_n(\lambda_y,\lambda)(\lambda-\lambda_x)(\lambda-\lambda_y)\,d\lambda\right|$$
$$\leqslant n\left[\int K_n^2(\lambda_x,\lambda)(\lambda-\lambda_x)^2\,d\lambda\int K_n^2(\lambda_y,\lambda)(\lambda-\lambda_y)^2\,d\lambda\right]^{1/2}$$

Now the arguments similar to those used in the proof of Lemmas 2 and 5 yield the estimate

$$\left|\frac{d}{dt}K\left(\lambda_0+\frac{x-tx}{n},\lambda_0+\frac{y-tx}{n}\right)\right|$$
$$\leqslant\frac{C}{n}|x|\left(\psi_n^2(\lambda_x)+\psi_{n-1}^2(\lambda_x)+\psi_n^2(\lambda_y)+\psi_{n-1}^2(\lambda_y)+\frac{|x-y|^2}{n}+e^{-nad/2}\right)$$

Combining this estimate with (3.25), we obtain (3.34).

Lemma 8. Let $p_2^{(n)}(\lambda_1, \lambda_2)$ be specified by (1.18) for $l=2$. Then uniformly in n

$$\int_{-1}^{1} \frac{p_2^{(n)}(\lambda_0 + x/n, \lambda_0)}{x^2}\, dx \leqslant C \tag{3.35}$$

Proof. Consider

$$W = \left\langle \prod_{i=2}^{n} \left| 1 - \frac{1}{n^2(\lambda_i - \lambda_0)^2} \right| \right\rangle$$

where symbol $\langle \ldots \rangle$ was defined in (2.11). By the Schwarz inequality W^2 is bounded from above by the product of integrals

$$Z_n^{-1} \int \prod_{2 \leqslant j < k \leqslant n} (\lambda_i - \lambda_j)^2 \prod_{2 \leqslant j \leqslant n} (\lambda_0 + \sigma - \lambda_j)^2$$
$$\times \exp\left\{ -nV(\lambda_0) - n \sum_{j=2}^{n} V(\lambda_j) \right\}$$

for $\sigma = \pm 1/n$. In addition, $n(V(\lambda_0) - V(\lambda_0 + \sigma))$ is bounded in n due to (1.11b). This allows us to write the bound

$$W \leqslant C \cdot K_n^{1/2}\left(\lambda_0 + \frac{1}{n}, \lambda_0 + \frac{1}{n}\right) K_n^{1/2}\left(\lambda_0 - \frac{1}{n}, \lambda_0 - \frac{1}{n}\right) \leqslant C_1 \tag{3.36}$$

On the other hand, W can be written as

$$W = \left\langle \prod_{i=2}^{n} (\phi_1(\lambda_i) + \phi_2(\lambda_i)) \right\rangle$$
$$= \left\langle \prod_{i=2}^{n} \phi_2(\lambda_i) \right\rangle + \sum_{k=1}^{n-1} C_{n-1}^{k} \left\langle \prod_{i=2}^{k+1} \phi_1(\lambda_i) \prod_{i=k+2}^{n} \phi_2(\lambda_i) \right\rangle$$

where

$$\phi_1(\lambda) = \begin{cases} \dfrac{(1 - n^2(\lambda - \lambda_0)^2)^2}{n^2(\lambda - \lambda_0)^2}, & n\,|\lambda - \lambda_0| < 1 \\ 0, & \text{otherwise} \end{cases}$$

and

$$\phi_2(\lambda) = \begin{cases} 1 - n^2(\lambda - \lambda_0)^2, & n\,|\lambda - \lambda_0| < 1 \\ \dfrac{n^2(\lambda-\lambda_0)^2 - 1}{n^2(\lambda - \lambda_0)^2}, & \text{otherwise} \end{cases}$$

Since $0 \leqslant \phi_2(\lambda) \leqslant 1$, $\phi_1(\lambda) \geqslant 0$, and $\langle 1 \rangle = \rho_n(\lambda_0)$, we get from this representation

$$W \geqslant -\rho_n(\lambda_0) + (n-1) \int d\lambda\, \phi_1(\lambda) \left\langle \delta(\lambda_2 - \lambda) \exp\left\{ \sum_{i=3}^{n} \log \phi_2(\lambda_i) \right\} \right\rangle \tag{3.37}$$

By using the Jensen inequality and (1.2) we have

$$\begin{aligned} &\frac{\langle \delta(\lambda_2 - \lambda) \exp\{\sum_{i=3}^{n} \log \phi_2(\lambda_i)\} \rangle}{\langle \delta(\lambda_2 - \lambda) \rangle} \\ &\quad \geqslant \exp\left\{ \left\langle \delta(\lambda_2 - \lambda) \sum_{i=3}^{n} \log \phi_2(\lambda_i) [p_2^{(n)}(\lambda_0, \lambda)]^{-1} \right\rangle \right\} \\ &\quad = \exp\left\{ (n-2) \int \log \phi_2(\lambda')\, p_3^{(n)}(\lambda_0, \lambda, \lambda')\, d\lambda' [p_2^{(n)}(\lambda_0, \lambda)]^{-1} \right\} \end{aligned} \tag{3.38}$$

According to (1.18),

$$\begin{aligned} p_3^{(n)}(\lambda_0, \lambda, \lambda') = \frac{n^2}{(n-1)(n-2)} \bigg(&\frac{n-1}{n} \rho_n(\lambda')\, p_2^{(n)}(\lambda_0, \lambda) \\ &+ 2K_n(\lambda_0, \lambda)\, K_n(\lambda_0, \lambda')\, K_n(\lambda, \lambda') \\ &- \rho_n(\lambda_0)\, K_n^2(\lambda, \lambda') - \rho_n(\lambda)\, K_n^2(\lambda_0, \lambda') \bigg) \end{aligned} \tag{3.39}$$

Moreover, since $\log \phi_2(\lambda') \leqslant 0$ and

$$\begin{aligned} &2K_n(\lambda_0, \lambda)\, K_n(\lambda_0, \lambda')\, K_n(\lambda', \lambda) \\ &\quad \leqslant 2K_n^{1/2}(\lambda_0, \lambda_0)\, K_n^{1/2}(\lambda, \lambda)\, |K_n(\lambda_0, \lambda')| \cdot |K_n(\lambda', \lambda)| \\ &\quad \leqslant \rho_n(\lambda_0)\, K_n^2(\lambda', \lambda) + \rho_n(\lambda)\, K_n^2(\lambda_0, \lambda') \end{aligned}$$

we have

$$\begin{aligned} &\int d\lambda' \log \phi_2(\lambda')(2K_n(\lambda_0, \lambda)\, K_n(\lambda_0, \lambda')\, K_n(\lambda', \lambda) \\ &\qquad - \rho_n(\lambda_0)\, K_n^2(\lambda, \lambda') - \rho_n(\lambda)\, K_n^2(\lambda_0, \lambda')) \geqslant 0 \end{aligned}$$

Hence, taking into account that $\rho_n(\lambda)$ is bounded from above uniformly in n, we get

$$W \geqslant -\rho_n(\lambda_0) + (n+1)\int d\lambda\, \phi_1(\lambda)\, p_2^{(n)}(\lambda_0, \lambda)$$

$$\times \exp\left\{(n-1)\int \rho_n(\lambda')\log \phi_2(\lambda')\, d\lambda'\right\}$$

$$\geqslant -\rho_n(\lambda_0) + \int_{-1}^{1} \frac{(1-x^2)^2}{x^2}\, p_2^{(n)}\left(\lambda_0, \lambda_0 + \frac{x}{n}\right) dx$$

$$\times \exp\left\{-C\left(\int_0^1 |\log(1-y^2)|\, dy + \int_1^\infty \log(1-y^{-2})\, dy\right)\right\} \qquad (3.40)$$

From (3.40) and (3.36) it is easy to derive (3.35).

Lemma 9. Let the functions $A(x, y)$, $B(x, y)$ be defined for $|x|$, $|y| \leqslant n\delta$, $\delta = n^{-1}\log n$, and satisfy the conditions

$$A(x, y) \leqslant a(x) \leqslant C_0, \qquad \int_{-1}^{1} \frac{a^2(x)}{x^2}\, dx \leqslant C_1^2 \qquad (3.41)$$

$$\int |A(x, y)|^2\, dx \leqslant C_2^2 \qquad (3.42)$$

$$|A(x, x) - A(-x, -x)| \leqslant \varepsilon_1 |x| \qquad (3.43)$$

$$|A(x, y) - B(x, y)| \leqslant \varepsilon_2 b(x), \qquad \int_{-1}^{1} \frac{b^2(x)\, dx}{x^2} \leqslant C_3^2, \qquad |b(x)| \leqslant C_4 \qquad (3.44)$$

$$\int_{-n\delta}^{n\delta} |A(x, y) - B(x, y)|^2\, dx \leqslant C_5^2 \varepsilon_3^2 \qquad (3.45)$$

Then

$$\int_{-n\delta}^{n\delta} \prod_{j=1}^{l} \frac{dx_j}{x_j} \det \|A(x_j, x_k)\|_{j,k=1}^{l} \leqslant (lC)^{l/2} \qquad (3.46)$$

and

$$\left|\int_{-n\delta}^{n\delta} \prod_{j=1}^{l} \frac{dx_j}{x_j} (\det \|A(x_j, x_k)\|_{j,k=1}^{l} - \det \|B(x_j, x_k)\|_{j,k=1}^{l})\right| \leqslant \varepsilon l (lC)^{l/2} \qquad (3.47)$$

where $\varepsilon = n\delta(\varepsilon_1 + \varepsilon_2) + \varepsilon_3$ and C depend only on C_i, $i = 1, \ldots, 5$.

Proof. It suffices to proves estimates (3.46) and (3.47) for

$$A_0(x_j, x_k) = A(x_j, x_k)(1-\delta_{jk})$$

and

$$B_0(x_j, x_k) = B(x_j, x_k)(1-\delta_{jk})$$

Indeed, due to conditions (3.43) and (3.44), the following inequalities hold:

$$\left|\int_{-n\delta}^{n\delta} \frac{A(x,x)}{x}\,dx\right| \leqslant 2n\delta\varepsilon_1$$

and

$$\left|\int_{-n\delta}^{n\delta} \frac{B(x,x)}{x}\,dx\right| \leqslant 2n\delta\varepsilon_1 + 2\varepsilon_2(C_3 + C_4 \log n\delta)$$

and we can easily obtain (3.46) and (3.47) for general A and B from the respective bounds for A_0 and B_0.

Consider

$$F(t) = \int_{-n\delta}^{n\delta} \prod_{j=1}^{l} \frac{dx_j}{x_j} \det \|A_t(x_j, x_k)\|_{j,k=1}^{l}$$

where $A_t(x_j, x_k) = A_0(x_j, x_k) + t(B_0 - A_0)(x_j, x_k)$. To obtain (3.46) we have to estimate $|F(1)-F(0)|$. Therefore it suffices to estimate dF/dt. Differentiating $F(t)$, making respective permutations of columns and rows, and the same renumbering of variables, we obtain

$$\frac{dF}{dt} = l\int_{-n\delta}^{n\delta} \prod_{j=1}^{l} \frac{dx_j}{x_j} \det \|D_t(x_j, x_k)\|_{j,k=1}^{l}$$

where

$$D_t(x_1, x_k) = (A_0 - B_0)(x_1, x_k), \qquad D_t(x_j, x_k) = A_t(x_j, x_k), \qquad j \geqslant 2$$

Thus

$$l^{-1}\left|\frac{dF}{dt}\right| \leqslant \int_{-1}^{1} \prod_{j=1}^{l} \frac{dx_j}{|x_j|} \det \|D_t(x_j, x_k)\|_{j,k=1}^{l}|$$

$$+\int_{-n\delta}^{n\delta} \prod_{j=1}^{l} \frac{dx_j}{|x_j|} (1-\chi_1(x_j))\, |\det \|D_t(x_j, x_k)\|_{j,k=1}^{l}|$$

$$+\sum_{m=1}^{l} C_l^m \left[\int_{-n\delta}^{n\delta} \prod_{j=1}^{m} \frac{dx_j}{|x_j|} \chi_1(x_j) \right.$$

$$\times \prod_{j=m+1}^{l} (1-\chi_1(x_j)) \frac{dx_j}{|x_j|} |\det \|D_l(x_j, x_k)\|_{j,k=1}^{l}|$$

$$+\int_{-n\delta}^{n\delta} \prod_{j=1}^{m} \frac{dx_j}{|x_j|} (1-\chi_1(x_j))$$

$$\left. \times \prod_{j=m+1}^{l} \chi_1(x_j) \frac{dx_j}{|x_j|} |\det \|D_l(x_j, x_k)\|_{j,k=1}^{l}| \right] \tag{3.48}$$

where $\chi_1(x)$ is the indicator of the interval $(-1, 1)$. Let us estimate the last term

$$F_r^{(m)} = \int_{-n\delta}^{n\delta} \prod_{j=1}^{m} \frac{dx_j}{|x_j|} (1-\chi_1(x_j))$$

$$\times \prod_{j=m+1}^{l} \chi_1(x_j) \frac{dx_j}{|x_j|} |\det \|D_l(x_j, x_k)\|_{j,k=1}^{l}|$$

Other terms in the r.h.s. of (3.48) can be estimated similarly,

$$|F_l^{(m)}| \leqslant \left\{ \int_{-n\delta}^{n\delta} \prod_{j=1}^{m} \frac{dx_j}{|x_j|^{3/4}} (1-\chi_1(x_j)) \right.$$

$$\left. \times \prod_{j=m+1}^{l} \chi_1(x_j) \frac{dx_j}{|x_j|^2} |\det \|D_l(x_j, x_k)\|_{j,k=1}^{l}|^2 \right\}^{1/2}$$

$$\times \left\{ \int_{-n\delta}^{n\delta} \prod_{j=1}^{m} \frac{dx_j}{|x_j|^{5/4}} (1-\chi_1(x_j)) \prod_{j=m+1}^{l} \chi_1(x_j)\, dx_j \right\}^{1/2}$$

$$\leqslant \left\{ \int_{-n\delta}^{n\delta} \prod_{j=1}^{m} \frac{dx_j}{|x_j|^{3/4}} (1-\chi_1(x_j)) \right.$$

$$\left. \times \prod_{j=m+1}^{l} \chi_1(x_j) \frac{dx_j}{|x_j|^2} \prod_{j=1}^{l} \sum_{k=1}^{l} D_l^2(x_j, x_k) \right\}^{1/2} \cdot 2^{m+l/2} \tag{3.49}$$

Here we used the Schwartz inequality and then the Hadamard estimate for determinants. Now the r.h.s. of (3.49) can be rewritten as the sum of the integrals

$$I_{j_1 \cdots j_l} = \int_{-n\delta}^{n\delta} \prod_{j=1}^{m} \frac{dx_j}{|x_j|^{3/4}} (1 - \chi_1(x_j))$$
$$\times \prod_{j=m+1}^{l} \chi_1(x_j) \frac{dx_j}{|x_j|^2} D_l^2(x_1, x_{j_1}) \ldots D_l^2(x_l, x_{j_l})$$
$$\leqslant \int_{-n\delta}^{n\delta} \prod_{j=1}^{m} \frac{dx_j}{|x_j|^{3/4}} (1 - \chi_1(x_j)) \prod_{j=m+1}^{l} \chi_1(x_j) \frac{dx_j}{|x_j|^2}$$
$$\times D_l^2(x_1, x_{j_1}) \ldots D_l^2(x_m, x_{j_m})\, a_l^2(x_{m+1}) \ldots a_l^2(x_l) \tag{3.50}$$

with $a_l(x) = a(x) + t\varepsilon_3 b(x)$.

To estimate the last integral, we start by integrating with respect to the "free" variables, i.e., the variables that do not enter the set $(x_{j_1}, \ldots, x_{j_m})$. We use bound (3.42) for the integral with respect to x_1 and bounds (3.44) and (3.45) for integrals with respect to $x_2 \ldots, x_m$. If there are no free variables, then we use the inequality $D_l^2(x_m, x_{j_m}) \leqslant (C_0 + t\varepsilon_2 C_4)^2$, which makes the variable x_{j_m} free. Repeating this procedure, we end up either with the estimate

$$I_{j_1 \ldots j_l} \leqslant C^{l-1} \Delta_{1,j} \tag{3.51}$$

or with the estimate

$$I_{j_1 \ldots j_l} \leqslant \varepsilon_3^2 C^{l-1} \Delta_{i,j} \tag{3.52}$$

where $C = \max\{2(C_1 + \varepsilon_2 C_3),\ C_0 + \varepsilon_2 C_4,\ 2(C_2 + \varepsilon_3 C_5)\}$ and

$$\Delta_{ij} = \int_{-n\delta}^{n\delta} D_l^2(x_i, x_j)(1 - \chi_1(x_i))(1 - \chi_1(x_j)) \frac{dx_i}{|x_i|^{3/4}} \frac{dx_j}{|x_j|^{3/4}}$$

with some $i, j \leqslant m$. Regarding $D_l^2(x_i, x_j)$ as the kernel of an integral operator acting in $L_2(-n\delta, n\delta)$ and using the bound $\sup_{x_i} \int_{-n\delta}^{n\delta} D_l^2(x_i, x_j)\, dx_j$ for the norm of this operator and bounds (3.42) and (3.45), we obtain that

$$I_{j_1 \ldots j_l} \leqslant \varepsilon_3^2 C^l$$

Repeating a similar argument to estimate all the other terms in (3.48), we obtain (3.46) and (3.47).

Lemma 10. Let T_n be defined by (2.13). Then

$$|T_n(\lambda_0) - \rho_n(\lambda_0)| \leqslant \frac{C}{\log n} \tag{3.53}$$

Proof. We will prove the following bounds:

$$|E\{U_n(\lambda_0)\} - \tfrac{1}{2}sV'(\lambda_0)| \leqslant Csn^{-1/4}\log n \tag{3.54}$$

$$\langle \exp\{2U_n(\lambda_0)\}\rangle \leqslant C \tag{3.55}$$

$$\langle |U_n(\lambda_0) - E\{U_n(\lambda_0)\}|^2\rangle \leqslant C/n\delta \tag{3.56}$$

where $U_n(\lambda_0)$ is specified by (2.15). Assuming that these bounds hold, it is easy to prove (3.53) by using the Schwarz inequality and the elementary inequality $|e^x - 1| \leqslant |x|\,(e^x + 1)$.

To prove (3.54) rewrite $E\{U_n(\lambda_0)\}$ as

$$\begin{aligned} E\{U_n(\lambda_0)\} &= (n-1)\int u(\lambda)\,\rho_n(\lambda)\,d\lambda \\ &= \int ((n-1)\,u(\lambda) - \phi(\lambda))(\rho_n(\lambda) - \rho(\lambda))\,d\lambda \\ &\quad + \int \phi(\lambda)(\rho_n(\lambda) - \rho(\lambda))\,d\lambda + (n-1)\int u(\lambda)\,\rho(\lambda)\,d\lambda \end{aligned} \tag{3.57}$$

where $\phi(\lambda)$ is a differentiable function of the form

$$\phi(\lambda) = \begin{cases} (n-1)\dfrac{\lambda_0 + \delta_1 - \lambda}{2\delta_1}\ln\left(1 - \dfrac{s}{n\delta_1}\right) & \\ \quad + (n-1)\dfrac{\lambda - \lambda_0 + \delta_1}{2\delta_1}\ln\left(1 + \dfrac{s}{n\delta_1}\right), & |\lambda - \lambda_0| < \delta_1 \\ (n-1)\,u(\lambda), & \text{otherwise} \end{cases}$$

where $\delta_1 = n^{-1/4}$. Using the Proposition and Lemma 4, one can estimate the first integral I_1 in the r.h.s. of (3.57) as follows:

$$I_1 \leqslant Cn^{-1/4}\int_{|\lambda - \lambda_0| \leqslant \delta_1} ((n-1\,|u(\lambda)| + |\phi(\lambda)|)\,d\lambda \leqslant Csn^{-1/4}\log n \tag{3.58}$$

To estimate the second integral, we use inequality (1.8), according to which

$$\begin{aligned} \left|\int \phi(\lambda)(\rho_n(\lambda) - \rho(\lambda))\,d\lambda\right| &\leqslant Cn^{-1/2}\log^{1/2} n\,\|\phi'\|_2^{1/2}\,\|\phi\|_2^{1/2} \\ &= C\delta_1^{-1}n^{-1/2}\log^{1/2} n \end{aligned} \tag{3.59}$$

The last integral I_3 in the r.h.s. of (3.57) can be calculated by using a result from ref. 9 according to which, for any λ, $\rho(\lambda) \neq 0$,

$$\int \log |\lambda - \lambda'| \, \rho(\lambda') \, d\lambda' = \tfrac{1}{2} V(\lambda) + \text{const} \tag{3.60}$$

Thus we have

$$I_3 = \tfrac{1}{2}(n-1)(V(\lambda_0 + s/n) - V(\lambda_0)) + O((n\delta_1)^{-1}) \tag{3.61}$$

Relations (3.57)–(3.61) prove (3.54).

To prove (3.55), consider $f(t) \equiv \log\langle \exp\{tU_n(\lambda_0)\}\rangle$. Since $f(t)$ is a convex function,

$$f(2) \leqslant f(0) + 2f'(2) = \log \rho_n(\lambda_0) + 2f'(2) \tag{3.62}$$

In view of (2.16)

$$\begin{aligned} f'(2) &= (n-1) \int u(\lambda) \, R_{2n}(\lambda, \lambda) \, d\lambda \\ &= (n-1) \int u(\lambda) \, R_{0n}(\lambda, \lambda) \, d\lambda + (n-1) \int_0^2 dt \int u(\lambda) \, R'_{tn}(\lambda, \lambda) \, d\lambda \end{aligned} \tag{3.63}$$

where R_{tn} and R'_{tn} are specified by (2.20) and (2.23). According to (2.3), the second integral I_2 in (3.63) can be rewritten as

$$\begin{aligned} I_2 = (n-1) \int_0^2 dt \bigg[&\int u^2(\lambda) \, R_{tn}(\lambda, \lambda) \, d\lambda \\ &+ (n-1) \int u(\lambda) \, u(\lambda') \, R^2_{tn}(\lambda, \lambda') \, d\lambda \, d\lambda' \bigg] \\ \leqslant C \int_{n\delta}^{\infty} &\log^2 \left(1 + \frac{s}{x}\right) dx \\ &+ (n-1)^2 \int_0^2 dt \int u(\lambda) \, u(\lambda') \, R^2_{tn}(\lambda, \lambda') \, d\lambda \, d\lambda' \end{aligned} \tag{3.64}$$

where we have used (2.29) for $\lambda = \mu$ and (1.20), according to which $R_{0n}(\lambda, \lambda) \leqslant K_n(\lambda, \lambda) = \rho_n(\lambda)$, the boundedness of $\rho_n(\lambda)$, and (2.27). Regarding $R^2_{tn}(\lambda, \lambda')$ as a kernel of integral operator $\bar{R}$ in $L_2(\mathbf{R})$, one can estimate its norm as $\|\bar{R}\| \leqslant \max_\lambda\{\int R^2_{tn}(\lambda, \lambda') \, d\lambda' = n^{-1}$. Thus

$$(n-1)^2 \int u(\lambda)\, u(\lambda')\, R_{m}^2(\lambda, \lambda')\, d\lambda\, d\lambda'$$

$$\leqslant n \int u^2(\lambda)\, d\lambda$$

$$\leqslant C \int_{n\delta}^{\infty} \log^2(1+x^{-1})\, dx \leqslant \frac{C_1}{n\delta} \tag{3.65}$$

To estimate the first integral I_1 in the r.h.s. of (3.63) we use again (2.29). Then

$$I_1 = (n-1) \int u(\lambda)\, R_{0n}(\lambda, \lambda)\, d\lambda$$

$$= (n-1) \int u(\lambda)\, K_n(\lambda, \lambda)\, d\lambda - \frac{n-1}{\rho_n(\lambda_0)} \int u(\lambda)\, K_n^2(\lambda_0, \lambda)\, d\lambda$$

$$= \frac{s}{2} V'(\lambda_0) + O(n^{-1/4} \log n) + O\left(n \frac{\max |u(\lambda)|}{\rho_n(\lambda_0)} \int K_n^2(\lambda_0, \lambda)\, d\lambda\right)$$

$$= \frac{s}{2} V'(\lambda_0) + O((n\delta)^{-1}) \tag{3.66}$$

Here we have used (3.54) to calculate $(n-1) \int u(\lambda)\, K_n(\lambda, \lambda)\, d\lambda$. Relations (3.62)–(3.66) prove (3.55).

To prove (3.66), let us note that in view of (3.54) and (3.66),

$$E\{U_n(\lambda_0)\} = \left\langle \sum_{j=2}^{n} u(\lambda_j) \right\rangle_0 + O((n\delta)^{-1})$$

where $\langle \ldots \rangle_0$ denotes the expectation with respect to density (2.18). This expectation is related to the operation $\langle \ldots \rangle = E\{\delta(\lambda_0 - \lambda_1) \ldots\}$ as $\langle \ldots \rangle = \rho_n(\lambda_0) \langle \ldots \rangle_0$. Thus, to estimate the r.h.s. of (3.56), it is enough to estimate

$$\left\langle \left(\sum_{j=2}^{n} u(\lambda_j) - \left\langle \sum_{j=2}^{n} u(\lambda_j) \right\rangle_0 \right)^2 \right\rangle_0$$

$$\leqslant (n-1) \int u^2(\lambda)\, R_{0n}(\lambda, \lambda)\, d\lambda$$

$$-(n-1)(n-2) \int u(\lambda)\, u(\lambda')\, R_{0n}^2(\lambda, \lambda')\, d\lambda\, d\lambda'$$

Combining this inequality with estimate (3.64), we get (3.56). Lemma 10 is proved.

Lemma 11. Let $X(x)$, $x\in\mathbf{R}$, be a smooth enough and rapidly decaying function. Then

$$\int\sum_{j=1}^{l}\frac{dx_j}{x_j}\det\|X(x_j-x_k)\|_{j,k=0}^{l}=\frac{(i\pi)^l X^{l+1}(0)}{l+1}\cdot\frac{1-(-1)^{l+1}}{2}\tag{3.67}$$

where $x_0=0$.

Proof. By using the Fourier integral representation of $X(x)$, we can write

$$\det\|X(x_j-x_k)\|_{j,k=0}^{l}=\int\prod_{j=0}^{l}dp_j\,\hat{X}(p_j)\det\|\exp\{ix_j(p_j-p_k)\|_{j,k=0}^{l}$$

where $\hat{X}(p)$ is the Fourier transform of $X(x)$. This representation and the identity

$$\int\frac{e^{ipx}}{x}dx=i\pi\operatorname{sign}p\equiv i\pi\theta(p)$$

allow us to rewrite the integral in the l.h.s. of (3.64) as follows:

$$(i\pi)^l\int\prod_{j=0}^{l}dp_j\,\hat{X}(p_j)\begin{vmatrix}1 & 1 & 1 & \cdot & 1\\ \theta(p_1-p_0) & 0 & \cdot & \cdot & \theta(p_1-p_l)\\ \theta(p_2-p_0) & \theta(p_2-p_1) & 0 & \cdot & \theta(p_2-p_l)\\ \cdot & \cdot & \cdot & \cdot & \cdot\\ \theta(p_l-p_0) & \cdot & \cdot & \cdot & 0\end{vmatrix}$$

Let us compute the determinant in the domain $p_1,\dots,p_m<p_0$, $p_{m+1},\dots,p_l\geqslant p_0$. Without loss of generality we can assume that $p_1<p_2\dots<p_m<p_0<p_{m+1}<\dots<p_l$. Then the determinant will have the form

$$\cdot\begin{vmatrix}+1 & +1 & +1 & +1 & \cdot & \cdot & +1 & +1\\ -1 & 0 & -1 & -1 & \cdot & \cdot & -1 & -1\\ -1 & +1 & 0 & -1 & \cdot & \cdot & \cdot & -1\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ -1 & +1 & +1 & \cdot & \cdot & \cdot & -1 & -1\\ +1 & +1 & +1 & +1 & \cdot & \cdot & -1 & -1\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ +1 & +1 & +1 & \cdot & \cdot & +1 & +1 & 0\end{vmatrix}$$

Subtracting the first row from the lth, $(l-1)$th,..., $(l-m)$th ones and then the first column from the second,..., mth ones, we find that the determinant is equal to $(-1)^{l-m}$. Therefore the l.h.s. of (3.67) is equal to

$$(i\pi)^l \sum_{m=0}^{l} C_l^m \int dp_0\, \hat{X}(p_0) \left(\int_{-\infty}^{p_0} dp\, \hat{X}(p)\right)^m \left(\int_{p_0}^{\infty} \hat{X}(p)\, dp\right)^{l-m}$$

$$= \int dp_0\, \hat{X}(p_0) \left(2\int_{-\infty}^{p_0} dp\, \hat{X}(p) - X(0)\right)^l$$

$$= \frac{(i\pi)^l X^{l+1}(0)}{l+1} \cdot \frac{1-(-1)^{l+1}}{2}$$

Lemma 11 is proved.

4. DISCUSSION

Let us regard the set $\{\lambda_j^{(n)}\}_{j=1}^n$ of eigenvalues of random matrices as a point process, i.e., as the random counting measure

$$\nu_n(\Delta) \equiv nN_n(\Delta) = \sum_{\lambda_j^{(n)} \in \Delta} 1 \tag{4.1}$$

Keeping in mind that we are studying the asymptotic behavior of the eigenvalue statistics for large n, we can define this point process either by the system (1.4) of its marginal distributions or by its generating functional

$$\Phi_n[\phi] = E\left\{\exp\left[\int \phi(\lambda)\, \nu_n(d\lambda)\right]\right\} \tag{4.2}$$

defined on a suitable space of test functions $\phi(\lambda)$, $\lambda \in \mathbf{R}$. We use the simplest case of bounded piecewise continuous functions with a compact support. Then, by using (1.17), we find that

$$\Phi_n[\phi] = \det(1 - k_n[\phi]) \tag{4.3}$$

where $k_n[\phi]$ is the integral operator defined on the support σ_ϕ of ϕ by the kernel

$$k_n(\lambda, \mu)(1 - e^{\phi(\mu)}) \tag{4.4}$$

According to the Theorem, the "scaling" limit (1.9) of all marginal densities (1.4) is given by (1.9) for all unitary invariant ensembles defined by (1.1), (1.7), and (1.11). To find the same limit for the generating functional we have to replace the test function $\phi(\lambda)$ by $\phi_n(x) = \phi(x/n\rho_n(\lambda_0))$. Then

$$\Phi[\phi] \equiv \lim_{n\to\infty} \Phi_n[\phi_n] = \det(1 - Q_\phi) \tag{4.5}$$

where Q_ϕ is the integral operator defined on σ_ϕ by the formula

$$(Q_\phi f)(x) = \int_{\sigma_\phi} S(x-y)(1-e^{\phi(y)})\, f(y)\, dy, \qquad x \in \sigma_\phi \tag{4.6}$$

and $S(x)$ is defined in (1.10). These formulas contain in fact the same information as (1.9), saying that in our case the point process

$$\nu_{\lambda_0}^{(n)}(t) = \nu_n\left(\lambda_0, \lambda_0 + \frac{t}{n\rho_n(\lambda_0)}\right) \tag{4.7}$$

converges weakly as $n \to \infty$ to the random process defined by (4.5) and (4.6) or by (1.9).

Consider now the probability

$$R_n(\{\Delta_j\}_{j=1}^{l}) = \Pr\{\nu_n(\Delta_j) = 0, j = 1,..., l\} \tag{4.8}$$

that an ordered set of disjoint intervals $\Delta_i = (a_i, b_i)$ does not contain eigenvalues. Then arguments similar to those proving (4.3) imply that

$$R_n(\{\Delta_j\}_{j=1}^{l}) = \det(1 - K_{n\Delta}) \tag{4.9}$$

where $\Delta = \bigcup_{j=1}^{l} \Delta_j$ and $K_{n\Delta}$ is the integral operator defined on Δ by the kernel

$$\sum_{j=1}^{l} \chi_{\Delta_j}(\lambda)\, k_n(\lambda, \mu)\, \chi_{\Delta_j}(\mu)$$

Setting

$$\begin{aligned} a_i &= \lambda_0 + \frac{\alpha_i}{n\rho_n(\lambda_0)}, \qquad & b_i &= \lambda_0 + \frac{\beta_i}{n\rho_n(\lambda_0)} \\ \delta_i &= (\alpha_i, \beta_i), \qquad & \delta &= \bigcup_{j=1}^{l} \delta_j \end{aligned} \tag{4.10}$$

and using the Theorem, we obtain that

$$\lim_{n\to\infty} R_n(\{\Delta_j\}_{j=1}^{l}) = r(\delta) \tag{4.11}$$

where $r(\delta)$ is the Fredholm determinant of the kernel

$$\sum_{j=1}^{l} \chi_{\delta_j}(x)\, S(x-y)\, \chi_{\delta_j}(y) \tag{4.12}$$

We can also introduce the more general kernel

$$\sum_{j=1}^{l} \tau_j \chi_{\delta_j}(x)\, S(x-y)\, \chi_{\delta_j}(y) \tag{4.13}$$

for an arbitrary collection of real τ_j. Then if for an arbitrary collection $k=(k_1,..., k_l)$ of positive integers we consider the probability

$$R_n(\{\Delta_j\}_{j=1}^{l}, \{k_j\}_{j=1}^{l}) = \Pr\{\nu_n(\Delta_j) = k_j\}$$

its limit $r(\delta, k)$ is

$$r(\delta, k) = \frac{(-1)^k}{k_1! \cdots k_l!} \cdot \frac{\partial^{k_1 + \cdots + k_l}}{\partial \tau_1^{k_1} \cdots \partial \tau_l^{k_l}} r(\delta, \tau)\bigg|_{\tau_j = 1} \tag{4.14}$$

where $r(\delta, \tau)$ is the Fredholm determinant of the kernel (4.13).

The case $l=1$ of (4.8) and (4.11) determines[1] the limiting probability distribution of distances between nearest neighbor eigenvalues (spacings) lying in the $O(n^{-1})$ neighborhood of λ_0. Thus in the limit (4.10) the spacing probability distribution is the same for all ensembles satisfying the conditions of the Theorem. For the Gaussian case, formula (4.14) was obtained in ref. 18, where some other kernels were also considered and various connections of the determinant (4.13) to integrable systems and related topics are discussed.

We can also consider another asymptotic regime, making "windows" in the $O(n^{-1})$ neighborhood of different spectral points, i.e., considering the joint probability distribution of the counting functions $\nu_{\lambda_1}^{(n)}(t_1),..., \nu_{\lambda_k}^{(n)}(t_k)$ for distinct n-independent $\lambda_1, ..., \lambda_k$. Take for simplicity $k=2$. Then we have to consider generating functional (4.2) on functions

$$\phi(\mu) = \phi_1(n\rho_n(\lambda_1)(\mu - \lambda_1)) + \phi_2(n\rho_n(\lambda_2)(\mu - \lambda_2))$$

Inserting this $\phi(\mu)$ in (4.2) and using a result from ref. 9 according to which $p_2^{(n)}(\lambda_1, \lambda_2) \to \rho(\lambda_1)\, \rho(\lambda_2)$ as $n \to \infty$ for distinct n-independent λ_1 and λ_2 and the Theorem, we obtain

$$\lim_{n \to \infty} \Phi_n[\phi] = \Phi[\phi_1]\, \Phi[\phi_2]$$

where $\Phi[\phi]$ is defined by (4.5) and (4.6).

We conclude that in the "scaling" limit the local statistics eigenvalues lying in $O(n^{-1})$ neighborhoods of distinct spectral points are independent.

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